ADDITIVE UNIT REPRESENTATIONS IN ENDOMORPHISM RINGS AND AN EXTENSION OF A RESULT OF DICKSON AND FULLER

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Dedicated to T. Y. Lam on his 70th Birthday

Abstract. A module is called automorphism-invariant if it is invariant under any automorphism of its injective hull. Dickson and Fuller have shown that if R is a finite-dimensional algebra over a field $\mathbb{F}$ with more than two elements then an indecomposable automorphism-invariant right R-module must be quasi-injective. In this note, we extend and simplify the proof of this result by showing that any automorphism-invariant module over an algebra over a field with more than two elements is quasi-injective. Our proof is based on the study of the additive unit structure of endomorphism rings.

1. Introduction.

The study of the additive unit structure of rings has a long tradition. The earliest instance may be found in the investigations of Dieudonné on Galois theory of simple and semisimple rings [4]. In [6], Hochschild studied additive unit representations of elements in simple algebras and proved that each element of a simple algebra over any field is a sum of units. Later, Zelinsky [15] proved that every linear transformation of a vector space V over a division ring D is the sum of two invertible linear transformations except when V is one-dimensional over $\mathbb{F}_2$. Zelinsky also noted in his paper that this result follows from a previous result of Wolfson [14].

The above mentioned result of Zelinsky has been recently extended by Khurana and Srivastava in [8] where they proved that any element in the endomorphism ring of a continuous module M is a sum of two automorphisms if and only if $\mathrm{End}(M)$ has no factor ring isomorphic to the field of two elements $\mathbb{F}_2$. In particular, this means that, in order to check if a module M is invariant under endomorphisms of its injective

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hull $E(M)$, it is enough to check it under automorphisms, provided that $\mathrm{End}(E(M))$ has no factor ring isomorphic to $\mathbb{F}_2$. Recall that a module M is called *quasi-injective* if every homomorphism from a submodule L of M to M can be extended to an endomorphism of M. Johnson and Wong characterized quasi-injective modules as those that are invariant under any endomorphism of their injective hulls [7].

A module M which is invariant under automorphisms of its injective hull is called an *automorphism-invariant module*. This class of modules was first studied by Dickson and Fuller in [3] for the particular case of finite-dimensional algebras over fields $\mathbb{F}$ with more than two elements. They proved that if R is a finite-dimensional algebra over a field $\mathbb{F}$ with more than two elements then an indecomposable automorphism-invariant right R-module must be quasi-injective. And it has been recently shown in [11] that this result fails to hold if $\mathbb{F}$ is a field of two elements. Let us recall that a ring R is said to be of *right invariant module type* if every indecomposable right R-module is quasi-injective. Thus, the result of Dickson and Fuller states that if R is a finite-dimensional algebra over a field $\mathbb{F}$ with more than two elements, then R is of right invariant module type if and only if every indecomposable right R-module is automorphism-invariant. Examples of automorphism-invariant modules which are not quasi-injective, can be found in [5] and [13]. And recently, it has been shown in [5] that a module M is automorphism-invariant if and only if every monomorphism from a submodule of M extends to an endomorphism of M. For more details on automorphism-invariant modules, see [5], [9], [11], and [12].

The purpose of this note is to exploit the above mentioned result of Khurana and Srivastava in [8] in order to extend, as well as to give a much easier proof, of Dickson and Fuller's result by showing that if M is any right R-module such that there are no ring homomorphisms from $\mathrm{End}_R(M)$ into the field of two elements $\mathbb{F}_2$, then M_R is automorphism-invariant if and only if it is quasi-injective. In particular, we deduce that if R is an algebra over a field $\mathbb{F}$ with more than two elements, then a right R-module M is automorphism-invariant if and only if it is quasi-injective.

Throughout this paper, R will always denote an associative ring with identity element and modules will be right unital. We refer to [1] for any undefined notion arising in the text.

Results.

We begin this section by proving a couple of lemmas that we will need in our main result.

Lemma 1. *Let M be a right R-module such that* $\mathrm{End}(M)$ *has no factor isomorphic to* $\mathbb{F}_2$*. Then* $\mathrm{End}(E(M))$ *has no factor isomorphic to* $\mathbb{F}_2$ *either.*

Proof. Let M be any right R-module such that $\mathrm{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$ and let $S = \mathrm{End}(E(M))$. We want to show that S has no factor isomorphic to $\mathbb{F}_2$. Assume to the contrary that $\psi : S \to \mathbb{F}_2$ is a ring homomorphism. As $\mathbb{F}_2 \cong \mathrm{End}_{\mathbb{Z}}(\mathbb{F}_2)$, the above ring homomorphism yields a right S-module structure to $\mathbb{F}_2$. Under this right S-module structure, $\psi : S \to \mathbb{F}_2$ becomes a homomorphism of S-modules. Moreover, as $\mathbb{F}_2$ is simple as $\mathbb{Z}$-module, so is as right S-module. Therefore, $\ker(\psi)$ contains the Jacobson radical $J(S)$ of S and thus, it factors through a ring homomorphism $\psi' : S/J(S) \to \mathbb{F}_2$.

On the other hand, given any endomorphism $f : M \to M$, it extends by injectivity to a (non-unique) endomorphism $\varphi_f : E(M) \to E(M)$

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ E(M) & \xrightarrow{\varphi_f} & E(M). \end{array}$$

Now define $\eta : \mathrm{End}(M) \to \frac{S}{J(S)}$ by $\eta(f) = \varphi_f + J(S)$. It may be easily checked that η is a ring homomorphism. Clearly, then $\eta \circ \psi' : \mathrm{End}(M) \to \mathbb{F}_2$ is a ring homomorphism. This shows that $\mathrm{End}(M)$ has a factor isomorphic to $\mathbb{F}_2$, a contradiction to our hypothesis. Hence, $\mathrm{End}(E(M))$ has no factor isomorphic to $\mathbb{F}_2$. $\square$

Lemma 2. ([8]) *Let M be a continuous right module over any ring S. Then each element of the endomorphism ring $R = \mathrm{End}(M_S)$ is the sum of two units if and only if R has no factor isomorphic to $\mathbb{F}_2$.*

We can now prove our main result.

Theorem 3. *Let M be any right R-module such that* $\mathrm{End}(M)$ *has no factor isomorphic to $\mathbb{F}_2$, then M is quasi-injective if and only M is automorphism-invariant.*

Proof. Let M be an automorphism-invariant right R-module such that $\mathrm{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$. Then by Lemma 1, $\mathrm{End}(E(M))$

has no factor isomorphic to $\mathbb{F}_2$. Now by Lemma 2, each element of $\mathrm{End}(E(M))$ is a sum of two units. Therefore, for every endomorphism $\lambda \in \mathrm{End}(E(M))$, we have $\lambda = u_1 + u_2$ where u_1, u_2 are automorphisms in $\mathrm{End}(E(M))$. As M is an automorphism-invariant module, it is invariant under both u_1 and u_2, and we get that M is invariant under λ. This shows that M is quasi-injective. The converse is obvious. $\square$

Lemma 4. *Let R be any ring and S, a subring of its center $Z(R)$. If $\mathbb{F}_2$ does not admit a structure of right S-module, then for any right R-module M, the endomorphism ring* $\mathrm{End}(M)$ *has no factor isomorphic to $\mathbb{F}_2$.*

Proof. Assume to the contrary that there is a ring homomorphism $\psi : \mathrm{End}_R(M) \to \mathbb{F}_2$. Now, define a map $\varphi : S \to \mathrm{End}_R(M)$ by the rule $\varphi(r) = \varphi_r$, for each $r \in S$, where $\varphi_r : M \to M$ is given as $\varphi_r(m) = mr$. Clearly φ is a ring homomorphism since $S \subseteq Z(R)$ and so, the composition $\varphi \circ f$ gives a nonzero ring homomorphism from S to $\mathbb{F}_2$, yielding a contradiction to the assumption that $\mathbb{F}_2$ does not admit a structure of right S-module. $\square$

We can now extend the above mentioned result of Dickson and Fuller.

Theorem 5. *Let A be an algebra over a field $\mathbb{F}$ with more than two elements. Then any right A-module M is automorphism-invariant if and only if M is quasi-injective.*

Proof. Let M be an automorphism-invariant right A-module. Since A is an algebra over a field $\mathbb{F}$ with more than two elements, by Lemma 4, it follows that $\mathbb{F}_2$ does not admit a structure of right $Z(A)$-module and therefore $\mathrm{End}(M)$ has no factor isomorphic to $\mathbb{F}_2$. Now, by Theorem 3, M must be quasi-injective. The converse is obvious. $\square$

As a consequence we have the following

Corollary 6. *Let R be any algebra over a field $\mathbb{F}$ with more than two elements. Then R is of right invariant module type if and only if every indecomposable right R-module is automorphism-invariant.*

Corollary 7. *If A is an algebra over a field $\mathbb{F}$ with more than two elements such that A is automorphism-invariant as a right A-module, then A is right self-injective.*

It is well-known that a group ring $R[G]$ is right self-injective if and only if R is right self-injective and G is finite (see [2], [10]). Thus, in particular, we have the following

Corollary 8. *Let $K[G]$ be automorphism-invariant, where K is a field with more than two elements. Then G must be finite.*

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